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A SYSTEM OF AXIOMATIC SET THEORY—PART VI[62]

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16. The rôle of the restrictive axiom. Comparability of classes. Till now we tried to get along without the axioms Vc and Vd. We found that this is possible in number theory and analysis as well as in general set theory, even keeping in the main to the usual way of procedure.

For the considerations of the present section application of the axioms Vc, Vd is essential. Our axiomatic basis here consists of the axioms I–III, V*, Vc, and Vd. From V*, as we know,[63] Va and Vb are derivable. We here take axiom V* in order to separate the arguments requiring the axiom of choice from the others. Instead of the two axioms V* and Vc, as was observed in Part II, V** may be taken as well.[63]

An obvious consequence of axiom Vd is that there exists for every cardinal a higher one; indeed the class of subsets of a set a is of higher power than a,[64] and so the set representing that class (by Vd)—let us call it as usual the "power-set" of a—has a higher cardinal number than a. From V* and Vc (or from V**), we can infer that for every sequence of ordinals s there exists an ordinal which is at least as high as each one of the members of s. For, the sum of the members of s by V* and Vc is represented by a set which is a transitive set of ordinals and thus is itself an ordinal.[65] This ordinal, having each member of s as a subset, must be at least as high as every member of s.

As a consequence of the stated theorem, there are two alternatives concerning classes of ordinals: Every class of ordinals A is either represented by a set, and then there exists a numeration of it in the natural order, or there exists a one-to-one correspondence between the class of all ordinals and the class A, every segment of which is an ascending sequence. Namely, if there exists a sequence s of elements of A such that there is no ordinal number belonging to A which is higher than every member of s, then, by the theorem just proved an ordinal n exists which is not lower than any member of s and thus not lower than any element of A; hence n' is higher than each element of A, and so A is a subclass of n'. Consequently A is represented by a set of ordinals, and of this there exists a numeration in the natural order.[66] In the other case, there exists, for every sequence of elements of A, an element of A which is higher than every member of the sequence; then, by a theorem noted in Part V as a consequence of the general recursion theorem,[67] there exists a one-to-one correspondence such as asserted for the second alternative.

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[63] Cf. Part II, §4, p. 3.

[64] Cf. Part IV, §11, p. 137.

[65] See Part II, §5, lemma 3, p. 7 and pp. 8, 9.

[66] See Part IV, §12, p. 143.

[67] Cf. Part V, §13, p. 91.

Applying this result to the class of all cardinals, we are here able to exclude the first alternative. For, if there existed a set of all cardinals and a numeration of it, this would be a sequence having every cardinal as its member; hence, as proved just above, there would exist an ordinal m which is at least as high as every cardinal. But then the cardinal number of m would be a highest cardinal, whereas in fact, by the consequence of Vd stated above, there exists no highest cardinal. Thus it follows that there exists a one-to-one correspondence between the class of all ordinals and the class of all cardinals, even with the property that every segment of it is an ascending sequence.

At the same time it follows from this reasoning that for every sequence of cardinals there exists a cardinal which is higher than every member of the sequence. This ascendence of powers is a characteristic feature of the full Cantor set theory.

A further characterization of the situation results from an application of the strong theorem of transfinite recursion which, as we stated in Part V,[68] is derivable from the axioms V* and Vc. According to this theorem, as a consequence of Vd, there exists the function Ψ whose domain is the class of all ordinals and whose values are determined by the conditions that (1) $\Psi(0) = 0$, (2) for every ordinal n, $\Psi(n')$ is the power-set of $\Psi(n)$, (3) for every limiting number l, $\Psi(l)$ is the set representing the sum of the members of the l-segment of Ψ. Of this function we first state that each of its values is a transitive set. As a preparatory result we show that a set p whose elements are the subsets of a transitive set t is itself transitive. In fact, if $b \epsilon p$ and $a \epsilon b$, then $b \subseteq t$ and thus $a \epsilon t$; and since t is transitive, $a \subseteq t$, and thus $a \epsilon p$.

Now our assertion that every value of Ψ is a transitive set follows by transfinite induction: 0 is a transitive set; if $\Psi(n)$ is transitive, then the set $\Psi(n')$ whose elements are the subsets of $\Psi(n)$ is also transitive; and, for every limiting number l such that $\Psi(n)$ is transitive for the ordinals n lower than l, $\Psi(l)$ represents the sum of transitive sets and is therefore transitive.

Further, for every ordinal n we have $\Psi(n) \epsilon \Psi(n')$, and the cardinal number of $\Psi(n)$ is lower than that of $\Psi(n')$. If l is a limiting number and $n \epsilon l$, we have $\Psi(n) \subset \Psi(l)$.

From the stated properties of Ψ by means of transfinite induction it follows readily that for ordinal numbers m, n,

$$m \epsilon n \rightarrow \Psi(m) \epsilon \Psi(n),$$

$$m \epsilon n \rightarrow \Psi(m) \subset \Psi(n),$$

and for every ordinal number n,

$$n \subseteq \Psi(n), n \epsilon \Psi(n').$$

At the same time we get the result that the function Ψ is a one-to-one correspondence, and that if m is an ordinal lower than the ordinal n, then the cardinal number of $\Psi(m)$ is lower than that of $\Psi(n)$.

[68] Cf. §13, p. 94.

Now let Π be the sum of the values of Ψ, or in other words, the class whose elements are the sets which, for some ordinal n, are in $\Psi(n)$.

Concerning this class Π some remarkable facts are to be stated, which were first set forth by von Neumann in his paper 299*b*, of course in a form related to his axiomatic system.[69] We shall follow here, in the main, his method of reasoning.

Let us call an element of Π a "Π-set" and a subclass of Π a "Π-class." We immediately state: Every ordinal n is a Π-set, since $n \in \Psi(n')$. Each element of a Π-set is itself a Π-set; for Π, being the sum of transitive sets, is a transitive class. Hence the class represented by a Π-set is a Π-class. Also, conversely, it is true that every set which represents a Π-class is a Π-set. For the proof of this we use the notion of the *degree* of a Π-set.

By the principle of the least number, for every Π-set a there is a least ordinal n such that $a \in \Psi(n)$; we call this ordinal, which is uniquely determined by a, the "degree of the Π-set a." There exists, by the class theorem, the class of pairs $\langle a, n\rangle$ such that $a \mathrel{\eta} \Pi$ and n is the degree of a.

Concerning the degree of Π-sets we have the following simple facts: (1) If n is the degree of a Π-set a, then for every ordinal m not lower than n, $a \in \Psi(m)$; for $a \in \Psi(n)$, and $\Psi(n) \subsetneqq \Psi(m)$. (2) An ordinal which is the degree of a Π-set is always a successor; for it can be neither 0 nor a limiting number. (3) The degree of an element a of a Π-set b is lower than the degree of b. For the degree of b, as we know from (2), is a successor n', and from $b \in \Psi(n')$ follows $b \subsetneqq \Psi(n)$ and thus $a \in \Psi(n)$; so the degree of a is lower than n'. (4) The degree of an ordinal n is n'. Namely, since $n \in \Psi(n')$, n is of no higher degree than n'; that it is of no lower degree than n' follows by transfinite induction using the statements (2) and (3).

Now with the aid of the concept of degree we can prove our assertion that every set which represents a Π-class is a Π-set, i.e. every set of Π-sets is a Π-set; the reasoning is as follows: If c is a set of Π-sets, then, by V*, there exists a set s of ordinal numbers n such that there is an element of c whose degree is n, and by Vc the sum of the elements of s is represented by a set. This sum, being a transitive set of ordinals, is itself an ordinal m, and m is at least as high as each element of s; hence, if $a \in c$, the degree of a is not higher than m, so that $a \in \Psi(m)$. Thus we have $c \subsetneqq \Psi(m)$, $c \in \Psi(m')$; and so c is a Π-set, as was to be proved.

From this it results that a set is a Π-set if and only if it represents a Π-class, or, in other words, if its elements are Π-sets. This theorem has the following immediate consequences: (1) Every subset of a Π-set is a Π-set. (2) A set (c), having c as its only element, is a Π-set if and only if c is a Π-set. (3) A pair $\langle a, b\rangle$ is a Π-set if and only if its members a, b are Π-sets. (4) If a and b are Π-sets, then the set–sum $a \dotplus (b)$ is a Π-set.

Now, using the proven theorem and the stated corollaries, the following result can be easily verified: If we start from a manifold of sets and classes with rela-

[69] Our class Π corresponds to von Neumann's "Bereich Π" (loc. cit. p. 237), our Ψ to his function ψ (p. 236); however, our definition of Ψ is somewhat simpler than von Neumann's definition of ψ.

tions ϵ, η satisfying the axioms I–III, V*, Vc, and Vd, then the assertions of these axioms remain valid if we reduce the range of sets and classes to that of Π-sets and Π-classes. Thus the reduced manifold with the original relations ϵ, η constitutes again a model for the axioms I–III, V*, Vc, and Vd. Let us call it the "Π-model."

Remark. Of course by this definition the Π-model is not independently determined, but merely relative to an original system of sets and classes with relations ϵ, η.

As an obvious statement about the Π-model we have: If the original system of sets and classes satisfies the axiom of infinity VI, then this axiom is likewise satisfied by the Π-model.

Further we observe that, the restrictive axiom VII is at all events satisfied by the Π-model. For, if A is a non-empty Π-class, then among the ordinal numbers which are the degrees of the elements of A there is a least one m; and if b is an element of A whose degree is m, every element of b has a degree lower than m; thus there cannot exist a common element of A and b. So the assertion of axiom VII holds for the Π-model.

On the other hand, upon the assumption of axiom VII we can infer that every set is a Π-set and hence the system of all sets and classes is identical with the Π-model. Indeed, since every set of Π-sets, by the theorem proved just above, is itself a Π-set, a set which is not a Π-set must have an element which is not a Π-set. Thus if A is the class of sets which are not Π-sets and $b \; \eta \; A$, then there must be a common element of A and b. But from this by axiom VII it follows that A is empty. Hence Π is the class of all sets.

So we see that, on the basis of the axioms I–III, V*, Vc, and Vd, the fulfilment of axiom VII is a condition upon which and only upon which the class Π is identical with the class of all sets and thus the Π-model identical with the system of all sets and classes.

In connection with this result we mention the observation of Gödel that on the basis of the axioms just mentioned, with the axiom of infinity VI added, axiom VII can be derived from the weaker axiom VII* which asserts for every non-empty set a there exists an element b such that a and b have no common element.[70]

As a preliminary for this derivation we first prove by the axioms I–III, V*, Vc, and VI that *for every set its transitive closure*[71] *is represented by a set.* By the axioms I–III and V*, from which, as we know, Va is derivable, the iteration theorem holds. By this theorem and Vc, for any set a there exists the iterator on a of the function assigning to each set c the set representing the sum of the elements of c. This iterator is a function B having the class N of the finite ordinals as its domain. Since N, by a consequence of axiom VI, is represented by a set, the sum of the values of B, by V* and Vc, is also represented by a set. That sum however is easily shown to be the transitive closure of a; and so we have that the transitive closure of an arbitrary set a is represented by a set.

[70] This observation, mentioned already in Part II (§4, p. 6), was communicated to the author by Gödel in July 1939.

[71] Cf. Part IV, §11, p. 136.

Now, using the proven theorem we can infer VII from VII*, by proving with the aid of VII* that every set is a Π-set. In fact the assumption that there exists a set a which is not a Π-set can be seen to contradict VII* as follows: The set a, not being a Π-set, must have an element e which is not a Π-set. Let c be the set representing the transitive closure of a; then $e \in c$, and thus the class of those elements of c which are not Π-sets is not empty, and likewise the set q representing that class is not empty. Now each element of q, not being a Π-set, must have an element which is not a Π-set. On the other hand, since $q \subseteq c$ and c is transitive, every element of an element of q must be in c. So it follows that each element of q must have an element b which is in c but is not a Π-set; but such an element b, by the definition of q, must be in q. Thus q is a non-empty set, such that for any element r there exists a common element of r and q; and so the assertion of VII* does not hold.

In this proof we had to use implicitly the definition of the function Ψ, since the concept of a Π-set refers to it.

A more simple derivation of VII from VII* is possible if besides axiom VI also the axiom of choice is available; even the weakened form IV* of the axiom of choice, formulated in Part III[72] (which follows from IV and the iteration theorem), is sufficient for this purpose.

Indeed, using the axioms I–III, IV*, V*, and VI we can show that if there is an instance contradicting VII, then VII* cannot hold either, and so VII* entails VII. The reasoning is as follows: Let A be a class not satisfying VII, i.e. a non-empty class whose every element has an element belonging to A. Then, by IV* applied to the class of pairs $\langle p, q\rangle$ such that $q \in p$, there exists a function F with the domain N assigning to every finite ordinal an element of A in such a way that, for each finite ordinal n, $F(n') \in F(n)$. The converse domain C of F is a subclass of A each of whose element has an elements belonging to C. Now from VI, as we know, it follows that the class N is represented by a set, and hence, by V*, the converse domain of F, i.e. the class C, is also represented by a set c. But c is non-empty, and for each element b of c there is a common element of b and C, and hence also of b and c. So the assertion of VII* is not satisfied.

Thus on the basis of the axioms I–III, IV*, V*, and VI, as well as of the axioms I–III, V*, Vc, Vd, and VI, axiom VII can be replaced by VII*.

Considering now the consequences of including the axiom of choice IV among our assumed axioms, so that we have the axioms I–V at our disposal, we are going to prove upon this axiomatic basis the *comparability* not only of any two Π-sets, but even *of any two* Π-*classes*. This in fact will result from the theorem that *every* Π-*class is of equal power with a class of ordinals*. The proof of this theorem by means of the axioms I–V can be given as follows: By the class theorem there exists the function K whose domain is the class of all ordinals and whose value for the ordinal n is the cardinal number of $\Psi(n)$. We are to show that the set-difference $\Psi(n') \dotdiv \Psi(n)$, i.e. the set of Π-sets of degree n', is of equal power with $\mathrm{K}(n') \dotdiv \mathrm{K}(n)$. For this purpose we distinguish the cases that n is finite or infinite. Since the class of subsets of a finite set is finite, it follows by complete

[72] See §10, p. 86.

induction that for every finite ordinal n the set $\Psi(n)$ is finite. Hence, if m is the cardinal number of $\Psi(n') \dotdiv \Psi(n)$, this also is the number attributable to $\Psi(n') \dotdiv \Psi(n)$, and we have $\mathrm{K}(n) + m = \mathrm{K}(n')$. From this equation on the other hand it follows that m is the number attributable to $\mathrm{K}(n') \dotdiv \mathrm{K}(n)$. Thus, if the ordinal n is finite,

$$(1) \qquad \Psi(n') \dotdiv \Psi(n) \sim \mathrm{K}(n') \dotdiv \mathrm{K}(n).$$

Let now n be infinite; then $\Psi(n)$, whose elements include the finite ordinals, is infinite; and since the cardinal number of $\Psi(n')$ is higher than that of $\Psi(n)$, we have

$$\Psi(n') \dotdiv \Psi(n) \sim \Psi(n'),$$

$$\mathrm{K}(n') \dotdiv \mathrm{K}(n) \sim \mathrm{K}(n'),$$

and by the definition of K: $\Psi(n') \sim \mathrm{K}(n')$. Hence (1) holds also for an infinite ordinal n. So the relation (1) has been proved for all ordinals n.

Now by the class theorem there exists a class of pairs $\langle n, c\rangle$ such that n is an ordinal and c is a one-to-one correspondence between the sets $\Psi(n') \dotdiv \Psi(n)$ and $\mathrm{K}(n') \dotdiv \mathrm{K}(n)$. The domain of this class of pairs in virtue of the relation (1), proved to hold for every ordinal, is the class of all ordinals. Hence by the axiom of choice there is a function F assigning to every ordinal n a one-to-one correspondence between $\Psi(n') \dotdiv \Psi(n)$ and $\mathrm{K}(n') \dotdiv \mathrm{K}(n)$. The sum S of the elements of the converse domain of F is a one-to-one correspondence. Indeed, if m, n are ordinals and m is lower than n, then the elements of $\Psi(m') \dotdiv \Psi(m)$, being Π-sets of degree m', all are different from the elements of $\Psi(n') \dotdiv \Psi(n)$, which are of degree n'; thus the sets $\Psi(m') \dotdiv \Psi(m)$, and $\Psi(n') \dotdiv \Psi(n)$ have no common element; but neither do $\mathrm{K}(m') \dotdiv \mathrm{K}(m)$ and $\mathrm{K}(n') \dotdiv \mathrm{K}(n)$ have common elements, since m' is not higher than n, and so $\Psi(m') \subseteq \Psi(n)$, and consequently $\mathrm{K}(m') \subseteq \mathrm{K}(n)$.

The domain of S, being the sum of the sets $\Psi(n') \dotdiv \Psi(n)$, is the class Π; for, the degree of every Π-set, as we stated, is a successor, and so every Π-set is in the sum of the sets $\Psi(n') \dotdiv \Psi(n)$. As to the converse domain of S, we could show without difficulty that it is the class of all ordinals. But here it is sufficient to use that its elements are ordinals. By this we have that there is a one-to-one correspondence between Π and a class of ordinals, and hence every subclass of Π, i.e. every Π-class, is of equal power with a class of ordinals, as was to be proved.

Now combining this result with the alternative on classes of ordinals stated in the beginning of this section, we infer that every Π-class is of equal power either with an ordinal number or with the class of all ordinals. In the first case, by Vb or by V*, the Π-class is represented by a set, which is a Π-set. Of course the class Π cannot be represented by a set, for otherwise the class of all ordinals, being a subclass of Π, would also be represented by a set. So *there is a one-to-one correspondence between* Π *and the class of all ordinals.*

We have now shown that any two Π-classes are comparable and that every Π-class is either represented by a Π-set or is of equal power with the class Π.

This result follows from the axioms I–V. If now axiom VII is added to these, then, as we know, we can infer that every class is a Π-class, and Π is the class of all sets. And so we come to state that any two classes are comparable, and that every class which is not represented by a set is of equal power with the class of all sets. By the last statement we have that a class is represented by a set if and only if it is of lower power than the class of all sets.

The contents of this assertion have been taken as an axiom by von Neumann in his axiomatic system.[73]

As a consequence of our results we note that the class of all cardinals is of equal power with the class of all sets. This, in fact, now follows from our former statement that there is a one-to-one correspondence between the class of all cardinals and that of all ordinals.

Another consequence is that *every class has a well-ordering;* indeed a well-ordering of a class C can be obtained from any one-to-one correspondence between C and a class of ordinals, and such a one-to-one correspondence, as we stated, exists for every class.

Let us further observe that by the given proof of the one-to-one correspondence existing between Π and the class of all ordinals a one-to-one correspondence has also been exhibited between the class of Π-sets of finite degree and the class of finite ordinals. For the proof of this one-to-one correspondence the axioms I–III and Va, or else I–III and VII, are sufficient. In fact the values of Ψ we have to deal with are merely those for finite ordinals, and for the definition of the function assigning to every finite ordinal n the value $\Psi(n)$ and for stating the properties of this function we can get along with the iteration theorem and ordinary complete induction.

17. Proofs of independence by means of models. The method of assigning to any model of the system of axioms I–III, V*, Vc, and Vd the corresponding Π-model, by which the results of the last section were derived, can also be used as a means for setting up models of independence. In fact, by deriving from the Π-model some more restricted models, we shall now show that in our axiom system I–VII none of the axioms VI, Vb, Vc, Vd, if excluded from the axiomatic basis, can be derived; or in other words, that each of these axioms is independent.

We start from the assumption—to be weakened afterwards—that there exists a model of the axiom system I–VI. The corresponding Π-model then satisfies the axioms I–VI, and also VII. Now we consider four subclasses of the class Π, which we shall denote by Π_0, Π_1, Π_2, Π_3. The definition can be given—denoting by ω, as usual, the least infinite cardinal, i.e. the set representing the class N of all finite ordinals, and by ω_1 the second infinite cardinal, i.e. the set representing the class Ω of all finite or enumerable ordinals—in the following way: Π_0 is the class represented by $\Psi(\omega)$, Π_1 is the class of those elements of $\Psi(\omega_1)$ which themselves as well as each element of their transitive closure are either finite or enumerable, Π_2 is the class represented by $\Psi(\omega + \omega)$, Π_3 is the class of those Π-sets which themselves as well as each element of their transitive closure are of lower power than $\Psi(\omega + \omega)$.

[73] It is the axiom IV*2*, cf. 299*2*, p. 225, and 299*5*, p. 675.

Each one of these classes $\Pi_r (r = 0, 1, 2, 3)$ determines a system of sets and classes by saying that the sets of that system are the elements of Π_r, and the classes of the system are the subclasses of Π_r; let us call it briefly the Π_r-system and let its sets and classes be called Π_r-sets and Π_r-classes.

Then we first state: (T_0) A Π-set is a Π_0-set if and only if it is finite and each of its elements is a Π_0-set. (T_1) A Π-set is a Π_1-set if and only if it is finite or enumerable and each of its elements is a Π_1-set. (T_2) A Π-set is a Π_2-set if and only if its degree is lower than $\omega + \omega$. (T_3) A Π-set is a Π_3-set if and only if it is of lower power than $\Psi(\omega + \omega)$ and each of its elements is a Π_3-set.

These statements are easy to verify, using the facts that (1) for every ordinal n the set $\Psi(n)$ is transitive and therefore every element of $\Psi(n)$ is also a subset of it, (2) for every finite ordinal n the set $\Psi(n)$ as also its elements are finite, (3) among the elements of a finite set of ordinals there is a highest ordinal, (4) for every enumerable sequence of elements of Ω there exists an ordinal belonging to Ω which is higher than each member of the sequence, and (5) the transitive closure of an element of a set c is a subclass of the transitive closure of c, the elements of c belong to the transitive closure of c, and each element of the transitive closure of c either is an element of c or belongs to the transitive closure of an elment of c.

From (T_0)–(T_3) we draw in particular the following obvious consequences (referring to the values 0, 1, 2, 3 of the subscript r): If a and b are Π_r-sets, then (a) and the set-sum $a \dotplus (b)$ are also Π_r-sets. A pair $\langle a, b\rangle$ is a Π_r-set if and only if a and b are Π_r-sets. Π_r is a transitive class, i.e. every element of a Π_r-set is a Π_r-set. Every subset (in the Π-model) of a Π_r-set is a Π_r-set. Now it is easy to verify that each of the Π_r-systems $(r = 0, 1, 2, 3)$ constitutes a model for the axioms I–III, IV, Va, and VII.

But with regard to the axioms Vb, c, d, and VI the four systems differ from one another. In fact let us consider the four Π_r-systems one by one with respect to the said axioms, applying for each of them the corresponding statement (T_r). We find the following results:

The Π_0-system satisfies Vb, c, d: (1) every class which is of equal power with a finite set is represented by a finite set, (2) the sum of the elements of a set, in case that the set itself as also its elements are finite, is a finite class, and therefore is represented by a finite set, (3) the class of the subsets of a finite set is represented by a finite set. The Π_0-system does not satisfy VI, but rather the following theorem incompatible with VI holds for it, that every Π_0-set is finite. (It is to be noticed however that there are infinite Π_0-classes.)

The Π_1-system satisfies VI and Vb, c: (1) the set ω representing the class N of finite ordinals is a Π_1-set, (2) every Π_1-class which is of equal power with a finite or an enumerable set is represented either by a finite or an enumerable set of Π_1-sets, (3) as a consequence of the theorem that the sum of the members of an enumerable sequence of enumerable sets is enumerable, which was proved in Part III,[74] the sum of the elements of a set, in case that the set and its elements are finite or enumerable, is a finite or enumerable class and therefore represented

[74] Cf. Part III, p. 87.

by either a finite or an enumerable set. The Π_1-system does not satisfy Vd, we rather have for it the theorem conflicting with Vd, that every Π_1-set is either finite or enumerable. (There are however non-enumerable Π_1-classes.)

The Π_2-system satisfies VI and Vc, d: (1) the set ω is of degree ω' and is therefore a Π_2-set, (2) every Π_2-set c has a degree n that is lower than $\omega + \omega$, the elements of the elements of c are Π-sets of degrees lower than n, thus the sum of the elements of c is a subclass of $\Psi(n)$ and hence is represented by an element of $\Psi(n')$, i.e. by a Π-set of degree n', but this then is a Π_2-set, so the sum of the elements of a Π_2-set is represented by a Π_2-set, (3) let n be the degree of a Π_2-set c, then n is lower than $\omega + \omega$, and $c \in \Psi(n)$, and since $\Psi(n)$ is transitive, $c \subsetneqq \Psi(n)$; hence every subset of c is a subset of $\Psi(n)$ and thus it is an element of $\Psi(n')$, so the class of subsets of c is a subclass of $\Psi(n')$ and therefore is represented by an element of $\Psi(n'')$, but every element of $\Psi(n'')$, since $n'' \in \omega + \omega$, is a Π_2-set; thus the class of subsets of a Π_2-set is represented by a Π_2-set. The Π_2-system does not satisfy Vb; in fact, there exists a one-to-one correspondence between the class of finite ordinals and the class of ordinals lower than $\omega + \omega$. The class of finite ordinals is represented by the Π_2-set ω; hence if Vb were valid for the Π_2-system, the class of ordinals lower than $\omega + \omega$ would be represented by a set and so $\omega + \omega$ would be a Π_2-set, but $\omega + \omega$, by a theorem of the preceding section,[75] has the degree $(\omega + \omega)'$, and so it is not a Π_2-set.

The Π_3-system satisfies VI and Vb, d: (1) the set ω is a Π_3-set, (2) a Π_3-class which is of equal power with a Π_3-set is represented by a Π-set, this set then is of lower power than $\Psi(\omega + \omega)$ and its elements are Π_3-sets, so it is a Π_3-set, (3) as we know from the proof of the existence of a one-to-one correspondence between the class Π and the class of all ordinals,[76] the cardinal number $\mathrm{K}(\omega + \omega)$ of $\Psi(\omega + \omega)$ is the sum of $\mathrm{K}(\omega)$ and the sets $\mathrm{K}(\omega + n') \div \mathrm{K}(\omega + n)$ with n being a finite ordinal, and so it is the lowest cardinal number which is higher than $\mathrm{K}(\omega + n)$ for each finite ordinal n, hence if a is a Π_3-set and c is its cardinal number, then, since c is lower than $\mathrm{K}(\omega + \omega)$, c is a subset of $\mathrm{K}(\omega + n)$ for some finite ordinal n, and so a is of equal power with a subset of $\Psi(\omega + n)$; from this it follows that the Π-set b which represents the class of the subsets of a is of not higher power than $\Psi(\omega + n')$, and thus of lower power than $\Psi(\omega + \omega)$, moreover its elements are Π_3-sets, hence it is itself a Π_3-set; so we have that the class of subsets of a Π_3-set is represented by a Π_3-set. Π_3 does not satisfy Vc. In fact, the enumerable Π-set whose elements are the sets $\Psi(\omega + n)$ with n being a finite ordinal, is a Π_3-set, but the set $\Psi(\omega + \omega)$ which represents the sum of the elements of that enumerable set, is not a Π_3-set.

It will appear that the manner in which we set up the four models of independence required more axiomatic assumptions than is really needed for that purpose. Indeed we started from a model for the system of all axioms I–VI, from which we proceeded to the corresponding Π-model. In this way we had the advantage that the classes Π_r could be defined as certain subclasses of Π. But by setting up each model separately, modifying either merely the way of defini-

[75] See p. 67.
[76] Cf. p. 70.

tion or also the model itself, we may be able to reduce the axiomatic basis, in particular to eliminate from it the axiom whose independence is to be shown by the model in question.

Let us briefly discuss some possible reductions of this kind which apply to the Π_r-systems with $r = 0, 1, 2$, or in other words to the models of independence relative to the axioms VI, Vd, and Vb. (The question here remains undecided if a model of the independence of axiom Vc can be established on the basis of the axioms I–IV, Va, b, d, and VI.)

The class Π_0 can be defined by first introducing the function Ψ_0, with the domain N, which is the iterator on 0 of the function whose domain is the class of finite sets and which assigns to every finite set a the power-set of a; Π_0 is the sum of the values of Ψ_0.

For this definition the axioms I–III and Va or else I–III and VII, from which the theory of finite sets can be obtained, as exhibited in Part II, are sufficient. These axioms also allow us to prove the properties of the Π_0-system, in particular that every Π_0-set is finite, that there exists a one-to-one correspondence between Π_0 and N, and as a consequence of these statements, that the Π_0-system constitutes a model for the system of axioms I–V and VII, however it does not satisfy VI.

For setting up the Π_1-system and proving that it satisfies all our axioms I–VII except Vd, the axioms I–III, IV, Va, b, and VI are sufficient. Namely on this basis, as was stated in Part III,[77] the axioms of the system of analysis are all available, either directly as axioms, or being derivable. By the system of analysis the theorem mentioned just above is provable that the sum of the members of an enumerable sequence of enumerable sets is enumerable; and from this it follows that the sum of the elements of an enumerable class of finite or enumerable sets is represented by an enumerable set.

Using this we can prove (by the considered axioms) that there exists a class of those sequences s whose domain is an element of Ω and which satisfy the conditions: $s(0) = 0$; $s(n')$ is, for an element n' of the domain of s, a finite or enumerable set of subsets of $s(n)$; and for a limiting number l which is in the domain of s, $s(l)$ is the set representing the sum of the members of the l-segment of s. Let us denote this class by Υ. Now the class Π_1 can be defined as the class of sets b such that, for some s belonging to Υ and some element n of the domain of s, $b \in s(n)$. And the degree of a Π_1-set (i.e. of an element of Π_1) is definable as the lowest of the ordinals n such that, for some sequence s belonging to Υ, n is in the domain of s and the Π_1-set is an element of $s(n)$. According to these definitions of Υ, Π_1, and "degree," the following can be proved:

(1) Every Π_1-set is either finite or enumerable. For the degree of a Π_1-set is a successor n', hence every Π_1-set is an element of some $s(n')$, with $s \,\eta\, \Upsilon$, and so it is a subset of $s(n)$; but every value of a sequence s belonging to Υ is either finite or enumerable as follows by transfinite induction[78] (using the above mentioned theorem of enumerability).

[77] Cf. Part III, p. 86.

[78] Cf. Part V, p. 89.

(2) Every element of Ω is a Π_1-set; indeed, as follows directly from the definition of Υ, every sequence s whose domain belongs to Ω and which assigns to each element k of its domain the value k, belongs to the class Υ.

(3) Every element of a Π_1-set b is again a Π_1-set, and of lower degree than b. For the degree of b being a successor n', we have $b \in s(n')$ for some s belonging to Υ; and so b is a subset of $s(n)$, hence if $a \in b$, we have $a \in s(n)$; thus a is a Π_1-set whose degree is at most n.

(4) The transitive closure of a finite or enumerable set a of Π_1-sets is a finite or enumerable class of Π_1-sets. Let C be the transitive closure of a, and let S be the class of finite or enumerable subsets of C. From our statement (3) we infer by means of complete induction that every element of C is a Π_1-set, and from the statement (1) it follows that the sum of the elements of any element of S is represented again by an element of S. Thus there exists the function F assigning to every set which belongs to S the set representing the sum of its elements and to every other set the null set, and the values of F all belong to S. Also $a \;\eta\; S$. So, if G is the iterator of F on a, every value of G is an element of S, i.e. a finite or enumerable subset of C, and also the sum of the values of G (since G is enumerable) is represented by an element of S. On the other hand, by the definition of the transitive closure it follows with the aid of complete induction that every element of C belongs to the sum of the values of G. Thus indeed C is a finite or enumerable class of Π_1-sets.

(5) Every finite or enumerable set a of Π_1-sets is a Π_1-set and its degree is the lowest of those ordinals which are successors and higher than the degree of any element of a. In fact, the sum of the degrees of the elements of a, being the sum of the elements of a finite or enumerable class of ordinals belonging to Ω, is represented by an element m of Ω; this is the lowest of those ordinals which are at least as high as the degree of any one element of a. Let C (as in the proof of (4)) be the transitive closure of a. By (4) the class C is finite or enumerable, and so every subclass of it is represented by a set; further the elements of such a set are Π_1-sets, and their degrees are not higher than m, as follows from our statement (3). So there exists an m'-sequence s assigning to every element n of m' the set of those elements of C which have a degree not higher than n. We have $s(0) = 0$; for every ordinal k' lower than m', every element of $s(k')$ is a set of Π_1-sets belonging to C which all, by our statement (3), are of degrees not higher than k and therefore are in $s(k)$, so that every element of $s(k')$ is a subset of $s(k)$, and thus $s(k')$ is a finite or enumerable set of subsets of $s(k)$; for every limiting number l lower than m', $s(l)$ is the set of those elements of C which have a degree lower than l, and so it is the sum of the members of the l-segment of s. Thus it follows that $s \;\eta\; \Upsilon$. Moreover, every element of a is in $s(m)$, so that $a \subseteq s(m)$. Now let t be the m''-sequence whose elements are those of s and also the pair $\langle m', (a)\rangle$. Then $t(m') = (a)$, $t(m) = s(m)$, so that $t(m')$ is a finite set of subsets of $t(m)$. Thus $t \;\eta\; \Upsilon$ and $a \in t(m')$. Hence a is a Π_1-set, whose degree is not higher than m', but neither can the degree of a be lower than m' since it must be higher than the degree of any element of a and it must be a successor. So m' is the degree of a, and it is the lowest ordinal which is a successor and is higher than any degree of an element of a.

From the statements (1)–(5) about Π_1, it is now quite easy to conclude that the Π_1-system satisfies all our axioms I–IV, Va, b, c, VI, VII, but not Vd.

As to the axiomatic basis of the exhibited theory of the Π_1-system, which is a variant of the theory of the Π-model given before, it is to be observed that instead of the axioms Vb and VI the axiom of the enumerable is sufficient here, and instead of Va axiom VII can be taken. On the other hand, if we keep Va, then instead of the axiom of the enumerable we can take axiom VI*, i.e. Fraenkel's generalization of Zermelo's axiom of infinity, which by the axioms I–III and Va is equivalent to the axiom of the enumerable.[79]

Thus for introducing the Π_1-system by means of the definition of the class Υ, and proving that it satisfies all our axioms I–VII with the exception of Vd, each of the following systems of axioms is sufficient:

I–IV, VII, and the axiom of the enumerable.

I–IV, Va, and the axiom of the enumerable.

I–IV, Va, and VI*.

For the proof of the existence of Υ and Π_1, as also of the statements (1)–(5) about Π_1, in each of three systems of axioms the axiom of choice IV could be replaced by the special axiom of choice IV_s; however, then the Π_1-system is not proved to satisfy IV, but only IV_s.

Remark. Upon the basis of all the axioms I–VI it is easy to prove, using the statements (1), (3), (5) of our preceding considerations and applying transfinite induction, that the class Π_1, as defined by means of the class Υ, is identical with the class of those elements of $\Psi(\omega_1)$ which themselves as well as every element of their transitive closure are either finite or enumerable; and further that the degree of a Π_1-set, as defined by means of the class Υ, is the same as the degree that this set has as a Π-set. Thus our later definition of Π_1 and of the degree of a Π_1-set is in agreement with the former definition.

Now we come to the question of setting up the Π_2-system without the axiom Vb. Here it is to be noticed that our axiom VI, in case Vb is not at our disposal, gives no way for proving any given infinite class to be represented by a set. So in order to establish the Π_2-system without Vb, we have to assume instead of VI a strengthened form of the axiom of infinity. Each of the following axioms "VI_1", "VI_2", (which are both derivable from VI, when the axioms I–III, Va, b are available), will suffice for this purpose:

VI_1. The class of all those sets which themselves as also every element of their transitive closure are finite is represented by a set.

VI_2. There exists a set c such that $0 \in c$, and if $a \in c$ and b is a set of subsets of a, then $b \in c$.

Indeed by each one of these axioms, in connection with I–III and Va, the class Π_0, whose existence, as we know, follows by means of the iteration theorem and the theorems on finiteness, can be shown to be represented by a set. Now let Ψ_1 be the iterator on this set of the function, existing by virtue of Vd, which assigns to every set its power-set. This function henceforth will be denoted briefly as the "power-set function." Then we can define Π_2, in accordance with

[79] Cf. Part III, pp. 68(bottom)–70, and p. 86.

our former definition, as the sum of the values of Ψ_1. A definition equivalent to this can be given by introducing—by analogy with the simple theory of types—the following concept of "type": A set a is said to be "of type n" if n is a finite ordinal, and $a \epsilon \Psi_1(n)$, and, for all $m \epsilon n$, a is not in $\Psi_1(m)$. A set is said to "have a type" if there exists a finite ordinal n such that the set is of type n; in this case obviously there is only one ordinal n such that the set is of type n, and this finite ordinal then is called "the type of" the set. Using the concept of type we can now define Π_2 as the class of sets which have a type.

The class of sets of type 0 is Π_0 and is represented by a set of type 1. For every finite ordinal n the class of sets of type n' is represented by the set-difference $\Psi_1(n') \dot{-} \Psi_1(n)$, which is a set of type n''. Further we have: A set is of type 0 if and only if it is a finite set of sets of type 0; a set not belonging to Π_0 is of type n' (for a finite ordinal n), if and only if every element of it is of a type not higher than n and at least one of its elements is of type n. As a consequence of this, every subset of a set of type n has a type not higher than n, and every element of a set of type n' has a type lower than n'. The infinite subclass N of Π_0 is represented by the set ω, which therefore is of type 1, and generally for every finite ordinal n, as follows by complete induction, the ordinal $\omega + n$ is of type n'. Hence there cannot be an ordinal higher than each of these ordinals $\omega + n$ and having a type, or in other words: every ordinal which belongs to Π_2 is lower than some ordinal $\omega + n$ with $n \epsilon \omega$.

From these statements it is now easy to infer that the Π_2-system, i.e. the system of Π_2-sets and Π_2-classes, by the new definition of Π_2 and on the basis of the axioms I–III, Va, Vd together with VI_1 or VI_2, satisfies all the axioms I–III, Va, c, d, VI, as also VI_1 and VI_2, whereas it does not satisfy Vb. Axiom VII is also satisfied by the Π_2-system. For let A be a non-empty Π_2-class; if the lowest type k occurring for an element of A is $\neq 0$, then for each element c of A whose type is k there cannot be a common element of A and c; on the other hand, if there are elements of A which are of type 0, then the class B of these elements is a non-empty subclass of Π_0, and since, as we know, the Π_0-system satisfies axiom VII, there is an element c of B which has no element in common with B, and hence A and c cannot have a common element either (since this, as a set of type 0, would belong to B); thus in both cases the assertion of VII is satisfied.

If the axiom of choice IV is included in the axiomatic basis, then it immediately follows that the Π_2-system satisfies also this axiom. Thus on the basis of the axioms I–IV, Va, Vd and either VI_1 or VI_2 we get a model for proving that in our system of axioms I–VII, axiom Vb is independent.

The model here considered has to be modified, if instead of one of the axioms VI_1, VI_2 we take Zermelo's axiom of infinity. We then are no more able to prove the existence of a set representing the class Π_0; but it follows from Zermelo's axiom of infinity, in connection with the axioms I–III and Va, that the sum of the values of the iterator on 0 of the function assigning to every set a the set (a) is represented by a set, let us call it ζ. Further let Ψ_0 be the iterator on 0 of the power-set function and Π_0 the sum of the values of Ψ_0, what is in accordance with the definition of Ψ_0 and Π_0 given before, upon a basis not including Vd.[80] By the

[80] Cf. p. 74.

corollary of the iteration theorem stated in Part II[81] combined with the class theorem, there exists a function Ψ^* assigning to every pair $\langle k, n \rangle$ of finite ordinals the value for n of the iterator of the power-set function on the set-sum $\zeta + \Psi_0(k'')$.

From the fact that ζ is transitive and that the power-set of a transitive set is transitive, it follows by complete induction, with respect first to k and then to n, that every value of Ψ^* is a transitive set; at the same time it results that, for finite ordinals k, n, we have $\Psi^*(\langle k, n \rangle) \subseteq \Psi^*(\langle k, n' \rangle)$ and $\Psi^*(\langle k, n \rangle) \subseteq \Psi^*(\langle k', n \rangle)$. Now we define the class Π^* as the sum of the values of Ψ^*; and for an element c of Π^*(a "Π^*-set"), we define its "type" to be the lowest finite ordinal n for which there exists a finite ordinal k such that $c \in \Psi^*(\langle k, n \rangle)$.

As immediate consequences of our definitions of Ψ^* and Π^* and "type", and of our statements on Ψ^* we have: (1) Every element as also every subset of a Π^*-set is a Π^*-set. (2) If a, b are Π^*-sets then also (a), (a, b), and the set-sum $a + (b)$ are Π^*-sets. (3) A pair $\langle a, b \rangle$ is a Π^*-set if and only if a and b are Π^*-sets. (Consequence of (1) and (2).) (4) The power-set of a Π^*-set is a Π^*-set. For, if $a \in \Psi^*(\langle k, n \rangle)$, then also $a \subseteq \Psi^*(\langle k, n \rangle)$; hence the power-set of a is a subset of $\Psi^*(\langle k, n' \rangle)$, hence it is in $\Psi^*(\langle k, n'' \rangle)$, and so it is a Π^*-set. (5) The sum of the elements of a Π^*-set is represented by a Π^*-set. For if $a \in \Psi^*(\langle k, n \rangle)$ and $b \in a$, then $b \in \Psi^*(\langle k, n \rangle)$ and also $b \subseteq \Psi^*(\langle k, n \rangle)$. Thus the sum of the elements of a is a subclass of $\Psi^*(\langle k, n \rangle)$ and hence, by Va, is represented by a set; this one, being a subset of $\Psi^*(\langle k, n \rangle)$, is an element of $\Psi^*(\langle k, n' \rangle)$ and so it is a Π^*-set. (6) The Π^*-sets of type 0 are the elements of Π_0. For every finite ordinal n, the elements of a Π^*-set of type n' are Π^*-sets of a type not higher than n.

From the statements (1)–(5) it appears that the Π^*-system, i.e. the system of Π^*-sets and Π^*-classes, satisfies our axioms I–III, Va, c, d. Zermelo's axiom of infinity and axiom VII are also satisfied by the Π^*-system; for the axiom of infinity this follows from the fact that $\zeta \in \Pi^*$, and for VII it can be inferred from the statement (6), by reasoning just as in the case of the Π_2-system.

As to the axiom of choice, there is again the possibility (as we had it for the Π_2-system) of immediately stating that it is satisfied by the Π^*-system, if this axiom is included in our axiomatic basis.

Likewise, if instead of IV the multiplicative axiom, in the form it was stated in Part II,[82] is assumed among our basic axioms, the assertion of this axiom can be proved to hold for the Π^*-system. Indeed, if s is a Π^*-set of non-empty sets, no two of them having a common element, then by the multiplicative axiom there exists a set c whose every element is in an element of s and which has just one element in common with each element of s. But such a set c, being a subset of the sum of the elements of the Π^*-set s, is again a Π^*-set, since Vc and Va hold for the Π^*-system.

Axiom Vb, of course, is not satisfied by the Π^*-system; there is even no Π^*-set

[81] See Part II, p. 12.

[82] Cf. Part II, p. 4. It is noted here as consequence (6) of our axioms. The proof here given is by IV and Vb; a better known proof is by IV and Va, c.

representing the class N, which, as we know, is of equal power with ζ. This can be shown by introducing the concept of the "ζ-degree" of a Π^*-set a; we define it as the lowest finite ordinal m which is the arithmetic sum of finite ordinals k, n such that $a \in \Psi^*(\langle k, n\rangle)$. By this definition every Π^*-set has a (uniquely determined) ζ-degree; the Π^*-sets of ζ-degree 0 are the elements of ζ, and for every finite ordinal m, which is $\neq 0$, the elements of a Π^*-set of ζ-degree m have a ζ-degree lower than m. From this, and the fact that 0 and 1 are the only ordinals which are elements of ζ and that every finite ordinal k' is in $\Psi_0(k'')$, it follows by complete induction that every finite ordinal m', considered as a Π^*-set, has the ζ-degree m. Thus there cannot be a Π^*-set of which every finite ordinal is an element; and so N is not represented by a Π^*-set.

As a result of our discussion of the Π^*-system we note: Upon the basis of our axioms I–III, Va, d together with Zermelo's axiom of infinity and the multiplicative axiom, the system of Π^*-sets and Π^*-classes constitutes a model for the axiom system consisting of the said axioms and also the axioms Vc and VII.

Now considering that the assumptions on the system of sets which are included in the said axiomatic basis are all derivable from Zermelo's original axioms and that on the other hand the original Zermelo axioms—if the concept of "definite Eigenschaft" is understood in our precise sense[83]—are all satisfied by the system of Π^*-sets, we can draw from our result the following consequence: If the original Zermelo axiom system is consistent, then no contradiction arises by adding the assumption that every set is a Π^*-set. Or what comes to the same: From the original Zermelo axioms, provided that they are consistent, no set which is not a Π^*-set can be proved to exist; in particular it is impossible to prove by them that there exists a set of which every finite ordinal, by our definition of "ordinal", is an element.

ZURICH

[83] Cf. Part I, p. 65, footnote 3.